Function group approach to unconstrained Hamiltonian Yang–Mills theory

Antti Salmela*
Theoretical Physics Division
Department of Physical Sciences
P.O. Box 64, 00014 University of Helsinki, Finland

Abstract

Starting from the temporal gauge Hamiltonian for classical pure Yang–Mills theory with the gauge group SU(2) a canonical transformation is initiated by parametrising the Gauss law generators with three new canonical variables. The construction of the remaining variables of the new set proceeds through a number of intermediate variables in several steps, which are suggested by the Poisson bracket relations and the gauge transformation properties of these variables. The unconstrained Hamiltonian is obtained from the original one by expressing it in the new variables and then setting the Gauss law generators to zero. This Hamiltonian turns out to be local and it decomposes into a finite Laurent series in powers of the coupling constant.

PACS: 11.15.Kc, 11.10.Ef, 02.20.Sv

*Email: Antti.Salmela@Helsinki.Fi

1 Introduction

An important and still open problem of quantum chromodynamics is to work out analytical predictions for the low-energy states of the theory. In order to make these predictions we need a proper quantum Yang–Mills theory which is valid in the low-energy regime. However, for many reasons it has turned out to be a difficult task to construct a useful physical Hamiltonian. One of the problems encountered is the implementation of Gauss's law in the Hamiltonian formalism. Up to this date, several methods have been developed to tackle it [1] – [13], and this paper aims to provide a novel method, which is motivated by Lie's theory of function groups and their canonical representations.

Usually one starts with an extended quantum Hamiltonian where the physical subspace consists of states that are annihilated by the Gauss law generators. In this paper, by contrast, the order of quantisation and constraining is reversed and Gauss's law is incorporated into the Hamiltonian formalism already at the classical level with the help of a suitable canonical transformation. Whenever one performs canonical transformations in a classical Hamiltonian gauge theory, one must choose the new variables in a way that makes their fundamental Poisson bracket relations compatible with the gauge algebra satisfied by the Gauss law generators. This is often done by the method of Abelianisation, where the Gauss law generators are multiplied by suitable matrices that transform them into mutually involutive canonical momenta. In this paper, however, the opposite strategy is followed and the generator algebra is taken as given. The generators are then parametrised with the minimum number of canonical variables in such a way that the gauge algebra is satisfied as a consequence of the fundamental Poisson brackets of the new variables. The remaining variables of the new set are finally constructed by following the logical steps implied by this parametrisation. The procedure is carried through for pure SU(2) Yang–Mills theory, but a generalisation to other Lie groups is discussed in the end.

The actual construction of the canonical transformation is done in several steps in section 2. The procedure is a bit lengthy, but I prefer to give a presentation where the underlying logic is made clear and where possibilities for modifications and generalisations are also offered. The final transformation is then used in the third section, where the unconstrained Hamiltonian is derived and expanded in a finite series involving both positive and negative powers of the coupling constant. The last section is devoted to conclusions. Throughout the paper I will use Einstein's summation convention with spatial and Lie algebra metrics normalised to positive unity. The generators of the SU(2) algebra are, as usual, taken to be $T_a = \frac{1}{2}\sigma_a$, where the σ_a's stand for the Pauli matrices.

2 Construction of the canonical transformation

2.1 Parametrisation of the Gauss law generators

We start with the temporal gauge ($A_0^a = 0$) Hamiltonian

$$H = \int \left(\frac{1}{2}\,\Pi_{ka}\Pi^{ka} + \frac{1}{4}\,F_{kl}^{a}\,F_{a}^{kl} \right) d^3\mathbf{x}, \tag{1}$$

where the field tensor F_{kl}^a is defined by

$$F_{kl}^a = \partial_l A_k^a - \partial_k A_l^a + g\,\varepsilon_{bc}{}^{a} A_k^b\,A_l^c.$$

The variables $A_k^a(\mathbf{x})$ and $\Pi_{ka}(\mathbf{x})$ are canonically conjugate, i.e., they satisfy the fundamental Poisson bracket relations

$$\{A_k^a(\mathbf{x}), \Pi_{lb}(\mathbf{y})\} = \delta_{kl}\, \delta^a{}_b\, \delta(\mathbf{x}-\mathbf{y}).$$

From these relations it follows that the Gauss law generators

$$G_a = \partial^k\, \Pi_{ka} - g\, \varepsilon_b{}^c{}_a A^{kb}\, \Pi_{kc} \tag{2}$$

obey the SU(2) algebra

$$\{G_a(\mathbf{x}), G_b(\mathbf{y})\} = -g\, \varepsilon_{ab}{}^c\, G_c(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}). \tag{3}$$

They also generate time-independent gauge transformations of the canonical variables as follows:

$$\begin{aligned} \{G_a(\mathbf{x}), A_k^b(\mathbf{y})\} &= -\delta_a{}^b\, \partial_k^{(x)} \delta(\mathbf{x}-\mathbf{y}) - g\, \varepsilon_{ca}{}^b\, A_k^c(\mathbf{y})\, \delta(\mathbf{x}-\mathbf{y}), \\ \{G_a(\mathbf{x}), \Pi_{kb}(\mathbf{y})\} &= -g\, \varepsilon^c{}_{ab}\, \Pi_{kc}(\mathbf{y})\, \delta(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{4}$$

The canonical equations of motion

$$\dot{A}_k^a(\mathbf{x}) = \frac{\delta H}{\delta \Pi_a^k(\mathbf{x})}, \qquad \dot{\Pi}_{ka}(\mathbf{x}) = -\frac{\delta H}{\delta A^{ka}(\mathbf{x})} \tag{5}$$

reproduce the dynamical Yang–Mills equations

$$\ddot{A}_k^a(\mathbf{x}) - \left[\delta^a{}_c\, \partial^l - g\, \varepsilon_{bc}{}^a A^{lb}(\mathbf{x})\right] F_{kl}^c(\mathbf{x}) = 0,$$

but not Gauss's law

$$G_a(\mathbf{x}) = 0.$$

However, the Gauss law generators are constants of motion, i.e.,

$$\dot{G}_a(\mathbf{x}) = 0$$

in the dynamics described by the equations (5). This property ensures that the implementation of Gauss's law can be done consistently with the Hamiltonian equations of motion. Unfortunately we cannot just use equation (2) to eliminate redundant coordinates in the limit $G_a \to 0$, because we do not know which coordinates to eliminate or how to deal with the canonical conjugates of these redundant variables.

The first stage in the function group approach consists of replacing the Gauss law generators with such canonical variables that will vanish in the limit when Gauss's law is put into force. At this point we recall that in Lie's work a function group is defined as a set of variables equipped with Poisson brackets that close on the set [14]. According to Lie, every function group can be transformed into a form where every variable either has a canonically conjugate counterpart in the set or its Poisson brackets with the remaining variables vanish. Applying this idea to the function group formed by the G_a's, we parametrise it with three canonical variables p_1, p_2 and q_2 as follows:

$$\begin{aligned} G_1 &= \sqrt{p_1^2 - p_2^2}\, \cos(g\, q_2) \\ G_2 &= -\sqrt{p_1^2 - p_2^2}\, \sin(g\, q_2) \\ G_3 &= p_2. \end{aligned} \tag{6}$$

It is easy to check that the SU(2) algebra relations (3) are satisfied if the Poisson brackets of the new variables are canonical, i.e., if

$$\{q_2(\mathbf{x}), p_2(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y})$$

and all the other brackets vanish. Conversely, we can invert this transformation and check that the variables

$$\begin{aligned} p_1 &= \sqrt{G_1^2 + G_2^2 + G_3^2} \\ p_2 &= G_3 \\ q_2 &= -\frac{2}{g} \arctan\left(\frac{\sqrt{G_1^2 + G_2^2} - G_1}{G_2}\right) \end{aligned} \tag{7}$$

satisfy the fundamental Poisson bracket relations by virtue of the algebra (3).

The parametrisation (6) is by no means the only possibility of defining a canonical representation, but it is one of the simplest with respect to the properties of the SU(2) algebra. Namely, equation (3) allows us to identify the G_a's with the basis vectors of the SU(2) algebra and the Poisson bracket with the commutator. We can now make use of the fact that for all semisimple Lie groups the Casimir operators together with the basis of the Cartan subalgebra span an Abelian subspace of the enveloping algebra. With higher-dimensional Lie groups this Abelian subspace can be augmented by Casimir operators of some lower-dimensional subalgebras. Since all canonical momenta must have vanishing Poisson brackets with each other, we see that the maximal set of momenta can be obtained from the maximal Abelian subspace of the enveloping algebra. This is the idea behind the transformation (7), where we now recognise p_1^2 as the Casimir operator of SU(2) and p_2 as the usual basis vector for the Cartan subalgebra. Once this choice has been made, the form of q_2 follows from the consistency of the canonical Poisson brackets with the algebra (3). However, the fundamental Poisson bracket relations do not determine the canonical conjugate of p_1 uniquely and we can thus leave the specific form of q_1 open at this stage. The next step is to extend the transformation (7) to the remaining variables.

2.2 Gauge-invariant variables

Let ξ_i stand for any new canonical variable not equal to those already fixed. By the requirement that the Poisson brackets between ξ_i and the members of the set $\{q_1, p_1, q_2, p_2\}$ vanish and with the help of the parametrisation (6) we see that

$$\{G_a(\mathbf{x}), \xi_i(\mathbf{y})\} = 0, \qquad a = 1, 2, 3, \tag{8a}$$

$$\frac{\delta \xi_i(\mathbf{y})}{\delta p_1(\mathbf{x})} = 0. \tag{8b}$$

In particular, equation (8a) means that all the remaining variables must be invariant under topologically trivial gauge transformations. Since we have already defined three non-gauge-invariant variables (q_1, q_2, p_2), they completely fix the gauge angles (modulo constant gauge transformations) in the new set of variables. The gauge-invariant fields must therefore be constructed by transforming the old variables into a gauge where

q_1, q_2 and p_2 are absent. Note that the term "gauge" does not imply neglecting any dynamical degrees of freedom at this stage, it only describes the way that the gauge-invariant variables of the new set are formed. In other words, the new variables consist of both gauge-dependent and gauge-invariant degrees of freedom. Although it may sound a little paradoxical, the gauge-invariant variables also satisfy a gauge condition due to the fact that, by construction, the gauge-dependent degrees of freedom have been transformed away. (This procedure is discussed in a more general context in Ref. [4].)

Let us begin with the elimination of q_2 and p_2. When these variables tend to zero, equation (6) shows that the components G_a tend to $\delta_{a1}\,p_1$. The intermediate variables $\widehat{A}_k^a$ and $\widehat{\Pi}_{ka}$ are then determined by the requirement that this property holds exactly, i.e.,

$$\begin{aligned} \widehat{A}_k^a &= (O_1)^a{}_b A_k^b - \frac{1}{2g}\,\varepsilon_{bc}{}^a \left(O_1\,\partial_k O_1^T\right)^{cb} \\ \widehat{\Pi}_{ka} &= (O_1)_a{}^b\,\Pi_{kb}, \end{aligned} \tag{9}$$

where the orthogonal matrix O_1 satisfies the relation

$$\widehat{G}_a = (O_1)_a{}^b G_b = \delta_{a1}\,p_1. \tag{10}$$

This is clearly fulfilled if we take

$$O_1 = \begin{pmatrix} \sqrt{1-(\frac{p_2}{p_1})^2}\cos(g\,q_2) & -\sqrt{1-(\frac{p_2}{p_1})^2}\sin(g\,q_2) & \frac{p_2}{p_1} \\ \frac{p_2}{p_1}\cos(g\,q_2) & -\frac{p_2}{p_1}\sin(g\,q_2) & -\sqrt{1-(\frac{p_2}{p_1})^2} \\ \sin(g\,q_2) & \cos(g\,q_2) & 0 \end{pmatrix} \tag{11a}$$

$$= \begin{pmatrix} \frac{G_1}{\sqrt{G_1^2+G_2^2+G_3^2}} & \frac{G_2}{\sqrt{G_1^2+G_2^2+G_3^2}} & \frac{G_3}{\sqrt{G_1^2+G_2^2+G_3^2}} \\ \frac{G_1G_3}{\sqrt{G_1^2+G_2^2}\sqrt{G_1^2+G_2^2+G_3^2}} & \frac{G_2G_3}{\sqrt{G_1^2+G_2^2}\sqrt{G_1^2+G_2^2+G_3^2}} & -\frac{\sqrt{G_1^2+G_2^2}}{\sqrt{G_1^2+G_2^2+G_3^2}} \\ -\frac{G_2}{\sqrt{G_1^2+G_2^2}} & \frac{G_1}{\sqrt{G_1^2+G_2^2}} & 0 \end{pmatrix}. \tag{11b}$$

It is interesting to note that the condition (10) falls in the category of Abelian gauges [15], where the gauge is partially fixed by diagonalising some homogeneously transforming object. In our case this object is the Gauss law generator $G = G^a\,T_a$, which is transformed into the direction of T_1. The residual U(1) gauge transformations are generated by T_1, and equation (10) then suggests that q_1 and p_1 are associated with this gauge freedom. More precisely, using the inverse formula (7) together with the properties (4) we can calculate the brackets

$$\begin{aligned} \{p_1(\mathbf{x}), \widehat{A}_k^a(\mathbf{y})\} &= -\delta^a{}_1\,\partial_k^{(x)}\delta(\mathbf{x}-\mathbf{y}) - g\,\varepsilon_{b1}{}^a\,\widehat{A}_k^b(\mathbf{y})\,\delta(\mathbf{x}-\mathbf{y}), \\ \{p_1(\mathbf{x}), \widehat{\Pi}_{ka}(\mathbf{y})\} &= -g\,\varepsilon^b{}_{1a}\,\widehat{\Pi}_{kb}(\mathbf{y})\,\delta(\mathbf{x}-\mathbf{y}), \end{aligned} \tag{12}$$

which prove that p_1 indeed generates U(1) rotations in the direction of T_1. On the other hand,

$$\{p_1(\mathbf{x}), \widehat{A}_k^a(\mathbf{y})\} = -\frac{\delta\widehat{A}_k^a(\mathbf{y})}{\delta q_1(\mathbf{x})}, \qquad \{p_1(\mathbf{x}), \widehat{\Pi}_{ka}(\mathbf{y})\} = -\frac{\delta\widehat{\Pi}_{ka}(\mathbf{y})}{\delta q_1(\mathbf{x})},$$

and combining these equations with the brackets (12) we get the following functional differential equations for the fields $\widehat{A}_k^a$ and $\widehat{\Pi}_{ka}$:

$$
\begin{aligned}
\frac{\delta \widehat{A}_k^a(\mathbf{y})}{\delta q_1(\mathbf{x})} &= \delta^a{}_1\, \partial_k^{(x)} \delta(\mathbf{x}-\mathbf{y}) + g\, \varepsilon_{b1}{}^a\, \widehat{A}_k^b(\mathbf{y})\, \delta(\mathbf{x}-\mathbf{y}), \\
\frac{\delta \widehat{\Pi}_{ka}(\mathbf{y})}{\delta q_1(\mathbf{x})} &= g\, \varepsilon^b{}_{1a}\, \widehat{\Pi}_{kb}(\mathbf{y})\, \delta(\mathbf{x}-\mathbf{y}).
\end{aligned}
$$

Given that these equations hold, it is then easy to see that new fields $\mathcal{A}_k^a$ and $\mathcal{\pi}_{ka}$ defined by

$$
\begin{aligned}
\mathcal{A}_k^a &= (O_2)^a{}_b \widehat{A}_k^b - \frac{1}{2g}\, \varepsilon_{bc}{}^a \left(O_2\, \partial_k O_2^T\right)^{cb} \\
\mathcal{\pi}_{ka} &= (O_2)_a{}^b\, \widehat{\Pi}_{kb}, \qquad (13) \\
O_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(g\, q_1) & -\sin(g\, q_1) \\ 0 & \sin(g\, q_1) & \cos(g\, q_1) \end{pmatrix} \qquad (14)
\end{aligned}
$$

are independent of q_1, i.e.,

$$
\frac{\delta \mathcal{A}_k^a(\mathbf{y})}{\delta q_1(\mathbf{x})} = 0, \qquad \frac{\delta \mathcal{\pi}_{ka}(\mathbf{y})}{\delta q_1(\mathbf{x})} = 0. \qquad (15)
$$

Combining the transformations (9) and (13) we can express the new variables in terms of the original fields A_k^a, Π_{ka} and the variables $\{q_1, p_1, q_2, p_2\}$. Employing formulas (4) and (6) together with the identity

$$
\{G_a(\mathbf{x}), q_1(\mathbf{y})\} = -\frac{\delta G_a(\mathbf{x})}{\delta p_1(\mathbf{y})}
$$

it is then a straightforward albeit rather lengthy exercise to check that the new variables are really gauge-invariant:

$$
\{G_a(\mathbf{x}), \mathcal{A}_k^b(\mathbf{y})\} = 0, \qquad \{G_a(\mathbf{x}), \mathcal{\pi}_{kb}(\mathbf{y})\} = 0.
$$

The requirement (8a) is thus satisfied, but this is not yet sufficient to make $\mathcal{A}_k^a$ and $\mathcal{\pi}_{ka}$ independent of p_1. Moreover, the new fields are redundant in number, because they satisfy the relation

$$
\mathcal{G}_a = \partial^k\, \mathcal{\pi}_{ka} - g\, \varepsilon_b{}^c{}_a \mathcal{A}^{kb}\, \mathcal{\pi}_{kc} = \delta_{a1}\, p_1, \qquad (16)
$$

which is actually more like a functional identity rather than a constraint, because it follows immediately from the transformations (9) and (13). Finally, $\mathcal{A}_k^a$ and $\mathcal{\pi}_{ka}$ are not canonical variables due to the fact that the gauge transformation matrices (11) and (14) depend on the original fields. Employing the fundamental Poisson brackets of the original variables and the gauge transformation properties (4) it is relatively straightforward to work out the brackets of $\mathcal{A}_k^a$ and $\mathcal{\pi}_{ka}$, but the calculations are lengthy. In fact, it becomes almost inevitable to use computer software capable of symbolic manipulations to perform these extensive calculations. Eventually we obtain the following result:

$$
\begin{aligned}
\{\mathcal{A}_k^a(\mathbf{x}), \mathcal{\pi}_{lb}(\mathbf{y})\} &= \frac{1}{p_1(\mathbf{y})} \Big[\delta^a{}_b\, \mathcal{\pi}_{l1}(\mathbf{y}) - \delta_{b1}\, \mathcal{\pi}_l^a(\mathbf{y})\Big]\, \partial_k^{(x)} \delta(\mathbf{x}-\mathbf{y}) \\
&\quad + \Big[\frac{g}{2\, p_1(\mathbf{y})} \left(\delta^a{}_1\, \varepsilon_{bc}{}^d + \delta_{b1}\, \varepsilon^a{}_c{}^d + \delta^d{}_1\, \varepsilon^a{}_{bc} + \delta_{c1}\, \varepsilon^a{}_b{}^d\right) \\
&\qquad \times \mathcal{A}_k^c(\mathbf{y}) \mathcal{\pi}_{ld}(\mathbf{y}) + \delta_{kl}\, \delta^a{}_b\Big]\, \delta(\mathbf{x}-\mathbf{y})
\end{aligned}
$$

$$+\left[\delta^a{}_1\,\partial^{(x)}_k - g\,\varepsilon_{c1}{}^a\mathcal{A}^c_k(\mathbf{x})\right]\frac{\delta\pi_{lb}(\mathbf{y})}{\delta p_1(\mathbf{x})}$$

$$+g\,\varepsilon^c{}_{1b}\,\pi_{lc}(\mathbf{y})\frac{\delta\mathcal{A}^a_k(\mathbf{x})}{\delta p_1(\mathbf{y})}, \qquad (17a)$$

$$\begin{aligned}
\{\mathcal{A}^a_k(\mathbf{x}),\mathcal{A}^b_l(\mathbf{y})\} &= \frac{1}{g}\,\varepsilon_1{}^{ab}\,\partial^{(y)}_l\left[\frac{1}{p_1(\mathbf{y})}\,\partial^{(x)}_k\delta(\mathbf{x}-\mathbf{y})\right]\\
&+\frac{1}{p_1(\mathbf{y})}\left[\delta^{ab}\,\mathcal{A}_{l1}(\mathbf{y})-\delta^b{}_1\,\mathcal{A}^a_l(\mathbf{y})\right]\partial^{(x)}_k\delta(\mathbf{x}-\mathbf{y})\\
&-\partial^{(y)}_l\left[\frac{1}{p_1(\mathbf{y})}\Big(\delta^{ab}\,\mathcal{A}_{k1}(\mathbf{y})-\delta^a{}_1\,\mathcal{A}^b_k(\mathbf{y})\Big)\,\delta(\mathbf{x}-\mathbf{y})\right]\\
&+\frac{g}{2\,p_1(\mathbf{y})}\Big(\delta^a{}_1\,\varepsilon^b{}_{cd}+\delta^b{}_1\,\varepsilon^a{}_{cd}+\delta_{d1}\,\varepsilon^{ab}{}_c+\delta_{c1}\,\varepsilon^{ab}{}_d\Big)\\
&\qquad\times\mathcal{A}^c_k(\mathbf{y})\mathcal{A}^d_l(\mathbf{y})\,\delta(\mathbf{x}-\mathbf{y})\\
&+\left[\delta^a{}_1\,\partial^{(x)}_k-g\,\varepsilon_{c1}{}^a\mathcal{A}^c_k(\mathbf{x})\right]\frac{\delta\mathcal{A}^b_l(\mathbf{y})}{\delta p_1(\mathbf{x})}\\
&-\left[\delta^b{}_1\,\partial^{(y)}_l-g\,\varepsilon_{c1}{}^b\mathcal{A}^c_l(\mathbf{y})\right]\frac{\delta\mathcal{A}^a_k(\mathbf{x})}{\delta p_1(\mathbf{y})}, \qquad (17b)
\end{aligned}$$

$$\begin{aligned}
\{\pi_{ka}(\mathbf{x}),\pi_{lb}(\mathbf{y})\} &= \frac{g}{2\,p_1(\mathbf{y})}\Big(\delta_{a1}\,\varepsilon_b{}^{cd}+\delta_{b1}\,\varepsilon_a{}^{cd}+\delta^d{}_1\,\varepsilon_{ab}{}^c+\delta^c{}_1\,\varepsilon_{ab}{}^d\Big)\\
&\times\pi_{kc}(\mathbf{y})\pi_{ld}(\mathbf{y})\,\delta(\mathbf{x}-\mathbf{y})-g\,\varepsilon^c{}_{1a}\,\pi_{kc}(\mathbf{x})\frac{\delta\pi_{lb}(\mathbf{y})}{\delta p_1(\mathbf{x})}\\
&+g\,\varepsilon^c{}_{1b}\,\pi_{lc}(\mathbf{y})\frac{\delta\pi_{ka}(\mathbf{x})}{\delta p_1(\mathbf{y})}. \qquad (17c)
\end{aligned}$$

As there are redundant coordinates in this set of variables, we should verify that these brackets are compatible with equation (16). Indeed, starting from the brackets (17) it is possible to derive the result

$$\{\mathcal{G}_a(\mathbf{x}),\mathcal{A}^b_k(\mathbf{y})\}=0,\qquad \{\mathcal{G}_a(\mathbf{x}),\pi_{kb}(\mathbf{y})\}=0,$$

which is consistent with equations (15) and (16). Our next task is to parametrise $\mathcal{A}^a_k$ and π_{ka} with new canonical variables in such a way that the relations (16) and (17) are satisfied.

2.3 Canonical variables

The elimination of the residual U(1) gauge degree of freedom with the transformation (13) was rather symbolic by nature, because the form of q_1 was not specified. The advantage of doing so is the fact that the Poisson brackets (17) now hold for all possible U(1) gauges and we can thus experiment with different gauge choices. Once a choice is made, its consistency with the brackets (17) then yields equations that determine the p_1-dependence of $\mathcal{A}^a_k$ and π_{ka}. The ingredients for choosing the gauge can be read off from the transformation formula (13). It is seen that the available objects fall in three categories: the pairs $(\widehat{A}^2_k,\widehat{A}^3_k)$ and $(\widehat{\Pi}_{k2},\widehat{\Pi}_{k3})$ form SO(2) doublets, the components $\widehat{\Pi}_{k1}$

are invariant and $\widehat{A}_k^1$ transforms as a photon. Although every gauge is possible from the physical point of view, yet in practice some gauges are not manifestly compatible with the brackets (17). For example, in the Coulomb gauge we should choose q_1 in such a way that the equation $\partial^k \mathcal{A}_k^1 = 0$ would hold as a functional identity, as indicated by equations (15). Therefore the Poisson brackets $\{\partial^k \mathcal{A}_k^1(\mathbf{x}), \partial^l \mathcal{A}_l^1(\mathbf{y})\}$ should also vanish identically, but according to the relations (17b) this is not the case. I cannot give a definite reason for this contradiction, but the canonical structure of the variables already fixed might be the cause and the problem could possibly be circumvented by adjusting q_1 and the definitions (7) suitably.

In the following calculations I have chosen the unitary gauge

$$\pi_{12}(\mathbf{x}) = 0.$$

Its consistency with the resulting identity $\{\pi_{12}(\mathbf{x}), \pi_{12}(\mathbf{y})\} = 0$ is obvious and it corresponds to defining q_1 by

$$q_1 = \frac{2}{g} \arctan\left(\frac{\sqrt{\widehat{\Pi}_{12}^2 + \widehat{\Pi}_{13}^2} - \widehat{\Pi}_{13}}{\widehat{\Pi}_{12}}\right). \tag{18}$$

Using formulas (4), (7) and (11b) it is a straightforward but lengthy calculation to verify that the fundamental Poisson bracket relations between q_1 and the variables $\{p_1, q_2, p_2\}$ indeed hold. Now the functional identities

$$\{\mathcal{A}_k^a(\mathbf{x}), \pi_{12}(\mathbf{y})\} = 0, \qquad \{\pi_{ka}(\mathbf{x}), \pi_{12}(\mathbf{y})\} = 0$$

combined with the brackets (17) give the following equations:

$$\begin{aligned}
\frac{\delta \mathcal{A}_k^a(\mathbf{x})}{\delta p_1(\mathbf{y})} &= -\frac{1}{g\, p_1(\mathbf{y})} \frac{\pi_{11}(\mathbf{y})}{\pi_{13}(\mathbf{y})} \delta^a{}_2\, \partial_k^{(x)} \delta(\mathbf{x}-\mathbf{y}) \\
&\quad + \frac{1}{\pi_{13}(\mathbf{y})} \left(\frac{\pi_{11}(\mathbf{y})}{p_1(\mathbf{y})} \varepsilon_{b2}{}^a \mathcal{A}_k^b(\mathbf{y}) - \frac{1}{g} \delta_{k1}\, \delta^a{}_2\right) \delta(\mathbf{x}-\mathbf{y}), \\
\frac{\delta \pi_{ka}(\mathbf{x})}{\delta p_1(\mathbf{y})} &= \frac{1}{p_1(\mathbf{y})} \frac{\pi_{11}(\mathbf{y})}{\pi_{13}(\mathbf{y})} \varepsilon^b{}_{2a}\, \pi_{kb}(\mathbf{y})\, \delta(\mathbf{x}-\mathbf{y}).
\end{aligned}$$

At first sight these equations look a bit frightening, but they turn out to be solvable with a reasonable effort. The solution can be written as a gauge transformation in the direction of T_2:

$$\begin{aligned}
\mathcal{A}_k^a &= (O_3)^a{}_b \left(\mathcal{Q}_k^b - \frac{1}{g} \delta_{k1}\, \delta^b{}_2 \frac{p_1}{\mathcal{P}_{11}} \cos\phi\right) - \frac{1}{2g} \varepsilon_{bc}{}^a \left(O_3\, \partial_k O_3^T\right)^{cb} \\
\pi_{ka} &= (O_3)_a{}^b\, \mathcal{P}_{kb}
\end{aligned} \tag{19}$$

with

$$O_3 = \begin{pmatrix} \sin\phi & 0 & -\cos\phi \\ 0 & -1 & 0 \\ -\cos\phi & 0 & -\sin\phi \end{pmatrix},$$

$$\sin\phi = \frac{\beta}{p_1}, \tag{20}$$

where $\mathcal{Q}^a_k$, $\mathcal{P}_{ka}$ and β are constants of integration, i.e.,

$$\frac{\delta \mathcal{Q}^a_k(\mathbf{x})}{\delta p_1(\mathbf{y})} = 0, \qquad \frac{\delta \mathcal{P}_{ka}(\mathbf{x})}{\delta p_1(\mathbf{y})} = 0, \qquad \frac{\delta \beta(\mathbf{x})}{\delta p_1(\mathbf{y})} = 0,$$

and

$$\mathcal{P}_{12}(\mathbf{x}) = \mathcal{P}_{13}(\mathbf{x}) = 0. \tag{21}$$

These constant fields fulfill both of the requirements (8), and therefore they should be adopted as new variables. The transformation formula is the inverse of equation (19), that is,

$$\begin{aligned} \mathcal{Q}^a_k &= (O_3)^a{}_b \left(\mathcal{A}^b_k + \frac{1}{g}\,\delta_{k1}\,\delta^b{}_2\,p_1\,\frac{\pi_{13}}{\pi_{11}^2+\pi_{13}^2} \right) - \frac{1}{2g}\,\varepsilon_{bc}{}^a \left(O_3\,\partial_k O_3^T \right)^{cb} \\ \mathcal{P}_{ka} &= (O_3)_a{}^b\,\pi_{kb}, \end{aligned} \tag{22}$$

where we now write the gauge angle as

$$\sin\phi = \frac{\pi_{11}}{\sqrt{\pi_{11}^2+\pi_{13}^2}}, \qquad \cos\phi = -\frac{\pi_{13}}{\sqrt{\pi_{11}^2+\pi_{13}^2}}. \tag{23}$$

Note that the matrix O_3 is both orthogonal and symmetric.

In order to proceed towards our final canonical transformation we must now find out whether the newest set of variables can be made canonical in accordance with the relation (16). Using equations (19) it is easy to see that the corresponding relation in the new variables reads

$$\partial^k\,\mathcal{P}_{ka} - g\,\varepsilon_b{}^c{}_a \mathcal{Q}^{kb}\,\mathcal{P}_{kc} = \delta_{a1}\,\beta. \tag{24}$$

The Poisson brackets are evaluated by inserting expressions (22) into the relations (17). Again this is a formidable calculation which requires extensive use of computer software. Here is the result:

$$\begin{aligned} \{\mathcal{Q}^a_k(\mathbf{x}), \mathcal{P}_{lb}(\mathbf{y})\} &= \delta_{kl}\,\delta^a{}_b\,\delta(\mathbf{x}-\mathbf{y}), \quad k \neq 1 \\ \{\mathcal{Q}^a_1(\mathbf{x}), \mathcal{P}_{11}(\mathbf{y})\} &= \delta^a{}_1\,\delta(\mathbf{x}-\mathbf{y}) \\ \{\mathcal{Q}^a_1(\mathbf{x}), \mathcal{P}_{lb}(\mathbf{y})\} &= -\frac{1}{\mathcal{P}_{11}(\mathbf{y})}\left[\delta^a{}_b \mathcal{P}_{l1}(\mathbf{y}) - \delta_{b1}\mathcal{P}^a_l(\mathbf{y})\right]\delta(\mathbf{x}-\mathbf{y}), \quad l \neq 1 \\ \{\mathcal{Q}^a_k(\mathbf{x}), \mathcal{Q}^b_l(\mathbf{y})\} &= 0, \quad k \neq 1, \quad l \neq 1 \\ \{\mathcal{Q}^a_k(\mathbf{x}), \mathcal{Q}^b_1(\mathbf{y})\} &= -\frac{1}{g}\,\varepsilon_1{}^{ab}\frac{1}{\mathcal{P}_{11}(\mathbf{y})}\,\partial^{(x)}_k\delta(\mathbf{x}-\mathbf{y}) \\ &\quad + \frac{1}{\mathcal{P}_{11}(\mathbf{y})}\left[\delta^{ab}\mathcal{Q}_{k1}(\mathbf{y}) - \delta^a{}_1\mathcal{Q}^b_k(\mathbf{y})\right]\delta(\mathbf{x}-\mathbf{y}), \quad k \neq 1 \\ \{\mathcal{Q}^a_1(\mathbf{x}), \mathcal{Q}^b_1(\mathbf{y})\} &= -\frac{1}{\mathcal{P}_{11}(\mathbf{y})}\left[\delta^a{}_1\,\mathcal{Q}^b_1(\mathbf{y}) - \delta^b{}_1\,\mathcal{Q}^a_1(\mathbf{y})\right. \\ &\qquad\qquad \left. + \frac{1}{g}\,\varepsilon_1{}^{ab}\,\frac{\beta(\mathbf{y}) - \partial^{(y)}_1\mathcal{P}_{11}(\mathbf{y})}{\mathcal{P}_{11}(\mathbf{y})}\right]\delta(\mathbf{x}-\mathbf{y}) \\ \{\mathcal{P}_{ka}(\mathbf{x}), \mathcal{P}_{lb}(\mathbf{y})\} &= 0 \end{aligned} \tag{25}$$

$$
\begin{aligned}
\{\beta(\mathbf{x}), \mathcal{Q}^a_k(\mathbf{y})\} &= -\delta^a{}_1\, \partial^{(x)}_k \delta(\mathbf{x}-\mathbf{y}) - g\, \varepsilon_{b1}{}^a\, \mathcal{Q}^b_k(\mathbf{y})\, \delta(\mathbf{x}-\mathbf{y}) \\
\{\beta(\mathbf{x}), \mathcal{P}_{ka}(\mathbf{y})\} &= -g\,(1-\delta_{k1})\, \varepsilon^b{}_{1a}\, \mathcal{P}_{kb}(\mathbf{y})\, \delta(\mathbf{x}-\mathbf{y}).
\end{aligned}
$$

Remember that $\mathcal{P}_{12} = \mathcal{P}_{13} = 0$ in these relations. It is now easy to construct the desired canonical fields. The brackets (25) suggest that we choose the pairs

$$
(\mathcal{Q}^1_1, \mathcal{P}_{11}), \qquad (\mathcal{Q}^a_k, \mathcal{P}_{ka}), \quad k \neq 1 \tag{26}
$$

as canonically conjugate variables. If we then solve the remaining variables from equation (24),

$$
\beta = \partial_1 \mathcal{P}_{11} + \sum_{k=2}^{3} \left(\partial_k \mathcal{P}_{k1} - g\, \varepsilon_b{}^c{}_1 \mathcal{Q}^b_k\, \mathcal{P}_{kc} \right) \tag{27a}
$$

$$
\mathcal{Q}^2_1 = -\frac{1}{g\,\mathcal{P}_{11}} \sum_{k=2}^{3} \left(\partial_k \mathcal{P}_{k3} - g\, \varepsilon_b{}^c{}_3 \mathcal{Q}^b_k\, \mathcal{P}_{kc} \right) \tag{27b}
$$

$$
\mathcal{Q}^3_1 = \frac{1}{g\,\mathcal{P}_{11}} \sum_{k=2}^{3} \left(\partial_k \mathcal{P}_{k2} - g\, \varepsilon_b{}^c{}_2 \mathcal{Q}^b_k\, \mathcal{P}_{kc} \right), \tag{27c}
$$

it turns out that all the Poisson brackets in the set (25) involving these variables follow from the fundamental brackets of the pairs (26). Unfortunately the variables (26), although gauge-invariant and canonical, are still not useful for implementing the Gauss law. The reason is equation (20), which shows that β tends to zero in the limit when p_1 vanishes. Looking at expression (27a), we see that it would be difficult to implement the requirement $\beta \to 0$ using the variables (26). A suitable canonical transformation is needed.

2.4 Canonical U(1) transformation

Passing to the variables (26), we have replaced the original SU(2) fields with a set of gauge-invariant canonical fields. However, the Poisson brackets (25) show that these variables have an inner U(1) symmetry, which is generated by β. Note that this symmetry has nothing to do with the original SU(2) symmetry since all the variables (26) and the generator β, defined by equation (27a), are gauge-invariant with respect to the generators G_a. Even so, we can apply the procedures of sections 2.1 – 2.3 also to this U(1) symmetry and choose β as a new canonical momentum variable, i.e.,

$$
p_3 = \partial_1 \mathcal{P}_{11} + \sum_{k=2}^{3} \left(\partial_k \mathcal{P}_{k1} - g\, \varepsilon_b{}^c{}_1 \mathcal{Q}^b_k\, \mathcal{P}_{kc} \right). \tag{28}
$$

The canonical conjugate of p_3 then determines the gauge angle associated with transformations generated by p_3, but again we leave the specific form of q_3 open at this stage. Since both q_3 and p_3 must have vanishing Poisson brackets with the remaining variables of the final canonical set, we conclude that the remaining variables must be functionally independent of q_3 and p_3. The elimination of q_3 can be done, as before, with a gauge transformation in the T_1-direction:

$$
\begin{aligned}
\widehat{Q}^a_k &= (O_4)^a{}_b \mathcal{Q}^b_k - \frac{1}{2g}\, \varepsilon_{bc}{}^a \left(O_4\, \partial_k O_4^T \right)^{cb} \\
\widehat{P}_{ka} &= (O_4)_a{}^b\, \mathcal{P}_{kb},
\end{aligned} \tag{29}
$$

where

$$O_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(g\,q_3) & -\sin(g\,q_3) \\ 0 & \sin(g\,q_3) & \cos(g\,q_3) \end{pmatrix}.$$

The Poisson brackets of the new variables follow from the algebra (25), the result being

$$\begin{aligned} \{\widehat{Q}^a_k(\mathbf{x}), \widehat{P}_{lb}(\mathbf{y})\} &= \delta_{kl}\,\delta^a{}_b\,\delta(\mathbf{x}-\mathbf{y}) + \left[\delta^a{}_1\,\partial^{(x)}_k - g\,\varepsilon_{c1}{}^a\widehat{Q}^c_k(\mathbf{x})\right]\frac{\delta\widehat{P}_{lb}(\mathbf{y})}{\delta p_3(\mathbf{x})} \\ &\quad + g\,\varepsilon^c{}_{1b}\,\widehat{P}_{lc}(\mathbf{y})\frac{\delta\widehat{Q}^a_k(\mathbf{x})}{\delta p_3(\mathbf{y})}, \\ \{\widehat{Q}^a_k(\mathbf{x}), \widehat{Q}^b_l(\mathbf{y})\} &= \left[\delta^a{}_1\,\partial^{(x)}_k - g\,\varepsilon_{c1}{}^a\widehat{Q}^c_k(\mathbf{x})\right]\frac{\delta\widehat{Q}^b_l(\mathbf{y})}{\delta p_3(\mathbf{x})} \\ &\quad - \left[\delta^b{}_1\,\partial^{(y)}_l - g\,\varepsilon_{c1}{}^b\widehat{Q}^c_l(\mathbf{y})\right]\frac{\delta\widehat{Q}^a_k(\mathbf{x})}{\delta p_3(\mathbf{y})}, \\ \{\widehat{P}_{ka}(\mathbf{x}), \widehat{P}_{lb}(\mathbf{y})\} &= -g\,\varepsilon^c{}_{1a}\,\widehat{P}_{kc}(\mathbf{x})\frac{\delta\widehat{P}_{lb}(\mathbf{y})}{\delta p_3(\mathbf{x})} + g\,\varepsilon^c{}_{1b}\,\widehat{P}_{lc}(\mathbf{y})\frac{\delta\widehat{P}_{ka}(\mathbf{x})}{\delta p_3(\mathbf{y})}. \end{aligned} \tag{30}$$

For the sake of clarity I have written down only those brackets that hold for the actual variables

$$(\widehat{Q}^1_1, \widehat{P}_{11}), \qquad (\widehat{Q}^a_k, \widehat{P}_{ka}), \quad k \neq 1.$$

In order to define variables independent of p_3 we must now specify the U(1) gauge by fixing q_3. I have chosen

$$q_3 = \frac{2}{g}\arctan\left(\frac{\sqrt{\mathcal{P}^2_{22} + \mathcal{P}^2_{23}} - \mathcal{P}_{23}}{\mathcal{P}_{22}}\right), \tag{31}$$

which corresponds to the identity

$$\widehat{P}_{22}(\mathbf{x}) = 0. \tag{32}$$

Making use of the brackets (25) it is possible to verify that q_3 and p_3 indeed satisfy

$$\{q_3(\mathbf{x}), p_3(\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y}),$$

while their brackets with the variables $\{q_1, p_1, q_2, p_2\}$ vanish due to the fact that both $\mathcal{Q}^a_k$ and $\mathcal{P}_{ka}$ meet the requirements (8). As before, the functional identities

$$\{\widehat{Q}^a_k(\mathbf{x}), \widehat{P}_{22}(\mathbf{y})\} = 0, \qquad \{\widehat{P}_{ka}(\mathbf{x}), \widehat{P}_{22}(\mathbf{y})\} = 0$$

lead to the equations

$$\begin{aligned} \frac{\delta\widehat{Q}^a_k(\mathbf{x})}{\delta p_3(\mathbf{y})} &= -\frac{1}{g\,\widehat{P}_{23}(\mathbf{y})}\,\delta_{k2}\,\delta^a{}_2\,\delta(\mathbf{x}-\mathbf{y}), \\ \frac{\delta\widehat{P}_{ka}(\mathbf{x})}{\delta p_3(\mathbf{y})} &= 0, \end{aligned}$$

whose solutions read

$$
\begin{aligned}
\widehat{Q}_k^a &= Q_k^a - \frac{1}{g}\,\delta_{k2}\,\delta^a{}_2\,\frac{p_3}{P_{23}} \\
\widehat{P}_{ka} &= P_{ka},
\end{aligned}
\tag{33}
$$

where Q_k^a and P_{ka} are constants of integration, i.e.,

$$
\frac{\delta Q_k^a(\mathbf{x})}{\delta p_3(\mathbf{y})} = 0, \qquad \frac{\delta P_{ka}(\mathbf{x})}{\delta p_3(\mathbf{y})} = 0.
$$

Now we choose these constants as new variables. Equation (28) then leads to the relation

$$
\partial_1\, P_{11} + \sum_{k=2}^{3} \left(\partial_k\, P_{k1} - g\,\varepsilon_b{}^c{}_1 Q_k^b\, P_{kc}\right) = 0,
\tag{34}
$$

which holds as a functional identity, implying that the new variables contain one redundant coordinate. The Poisson brackets of Q_k^a and P_{ka} are easily evaluated with the help of the relations (30). The result reads

$$
\begin{aligned}
\{Q_k^a(\mathbf{x}), P_{lb}(\mathbf{y})\} &= \delta_{kl}\,\delta^a{}_b\,\delta(\mathbf{x}-\mathbf{y}) - \delta_{k2}\,\delta^a{}_2\,\varepsilon^c{}_{1b}\,\frac{1}{P_{23}(\mathbf{y})} P_{lc}(\mathbf{y})\,\delta(\mathbf{x}-\mathbf{y}) \\
\{Q_k^a(\mathbf{x}), Q_l^b(\mathbf{y})\} &= 0, \qquad (k,a) \neq (2,2), \quad (l,b) \neq (2,2) \\
\{Q_k^a(\mathbf{x}), Q_2^2(\mathbf{y})\} &= -\frac{1}{g\,P_{23}(\mathbf{y})}\left[\delta^a{}_1\,\partial_k^{(x)} - g\,\varepsilon_{c1}{}^a\,Q_k^c(\mathbf{x})\right]\delta(\mathbf{x}-\mathbf{y}), \\
&\qquad (k,a) \neq (2,2) \\
\{Q_2^2(\mathbf{x}), Q_2^2(\mathbf{y})\} &= 0 \\
\{P_{ka}(\mathbf{x}), P_{lb}(\mathbf{y})\} &= 0,
\end{aligned}
\tag{35}
$$

showing that the pairs

$$
(Q_1^1, P_{11}), \qquad (Q_2^a, P_{2a}), \quad a = 1,3, \qquad (Q_3^a, P_{3a}), \quad a = 1,2,3
$$

are the most natural choice for final canonical variables. Solving equation (34) for the redundant coordinate,

$$
Q_2^2 = \frac{1}{g\,P_{23}}\left(\partial^k\, P_{k1} - g\,\varepsilon_b{}^c{}_1\, Q_3^b\, P_{3c}\right),
\tag{36}
$$

it is easy to see that all of the Poisson bracket relations (35) hold true. Our search for suitable canonical variables is now over.

2.5 Results

Starting from the original canonical fields (A_k^a, Π_{ka}) and passing through four sets of intermediate variables we have found the final canonical pairs

$$
\begin{aligned}
&(q_i, p_i), \qquad i = 1,2,3 \\
&(Q_1^1, P_{11}) \\
&(Q_2^a, P_{2a}), \quad a = 1,3 \\
&(Q_3^a, P_{3a}), \quad a = 1,2,3.
\end{aligned}
\tag{37}
$$

Equation (7) relates p_1, q_2 and p_2 to the original variables, and a formula for q_1 is obtained by combining equations (18), (9) and (11b). The momentum p_3 is most easily calculated by combining equations (20), (23), (13) and (9), while it takes successive applications of equations (31), (22), (13) and (9) to work out a formula for q_3. The remaining variables of the set (37) are then obtained by performing the transformations (33), (29), (22), (13) and (9) one after the other. Again the manipulations are so lengthy that computer assistance is required. Introducing the notation

$$||X|| := \sqrt{X_a X^a}$$

for the Lie algebra norm, the results can be written as follows:

$$q_1 = \frac{2}{g} \arctan \left(\frac{\sqrt{\widehat{\Pi}_{12}^2 + \widehat{\Pi}_{13}^2} - \widehat{\Pi}_{13}}{\widehat{\Pi}_{12}} \right), \tag{38a}$$

$$\widehat{\Pi}_{12} = \frac{1}{||G||} \left[\frac{G_3}{\sqrt{G_1^2 + G_2^2}} \left(G_1 \Pi_{11} + G_2 \Pi_{12} \right) - \sqrt{G_1^2 + G_2^2}\, \Pi_{13} \right]$$

$$\widehat{\Pi}_{13} = \frac{1}{\sqrt{G_1^2 + G_2^2}} \left(-G_2 \Pi_{11} + G_1 \Pi_{12} \right)$$

$$q_2 = -\frac{2}{g} \arctan \left(\frac{\sqrt{G_1^2 + G_2^2} - G_1}{G_2} \right) \tag{38b}$$

$$q_3 = \frac{2}{g} \arctan \left(\frac{\sqrt{\mathcal{P}_{22}^2 + \mathcal{P}_{23}^2} - \mathcal{P}_{23}}{\mathcal{P}_{22}} \right), \tag{38c}$$

$$\mathcal{P}_{22} = \frac{1}{||\mathcal{N}||} \varepsilon^{abc} G_a \Pi_{1b} \Pi_{2c}$$

$$\mathcal{P}_{23} = \frac{1}{||\mathcal{N}||} \frac{1}{||\Pi_1||} \left(\delta^{ad} \delta^{bc} - \delta^{ab} \delta^{cd} \right) G_a \Pi_{1b} \Pi_{1c} \Pi_{2d}$$

$$\mathcal{N}_a = \varepsilon_a{}^{bc} G_b \Pi_{1c}$$

$$p_1 = ||G|| \tag{38d}$$

$$p_2 = G_3 \tag{38e}$$

$$p_3 = \frac{G^a \Pi_{1a}}{||\Pi_1||} \tag{38f}$$

$$Q_k^a = \Omega^a{}_b A_k^b - \frac{1}{2g} \varepsilon_{bc}{}^a \left(\Omega\, \partial_k \Omega^T \right)^{cb} \tag{38g}$$

$$P_{ka} = \Omega_a{}^b \Pi_{kb}, \tag{38h}$$

where

$$\Omega_a{}^b = \left(O_4 O_3 O_2 O_1 \right)_a{}^b$$

$$
\begin{aligned}
&= \delta_{a1}\,\frac{1}{||\Pi_1||}\,\Pi_1^b + \delta_{a2}\,\frac{1}{||N||}\,\varepsilon^{bcd}\,\Pi_{2c}\Pi_{1d} \\
&\quad + \delta_{a3}\,\frac{1}{||\Pi_1||}\,\frac{1}{||N||}\left(\delta^{be}\,\delta^{cd} - \delta^{bc}\,\delta^{de}\right)\Pi_{1c}\Pi_{1d}\Pi_{2e}, \qquad (39) \\
N_a &= \varepsilon_a{}^{bc}\,\Pi_{1b}\Pi_{2c}.
\end{aligned}
$$

This transformation is singular when $||\Pi_1||$ or $||N||$ vanishes, corresponding to points where the gauge angles (23) and (38c) become ambiguous. These singularities are Gribov ambiguities [16] peculiar to unitary gauges, and it is well known that such ambiguities appear in almost every gauge [17].

When inverting the transformation (38) it should be noted that formula (38g) holds for variables of the set (37) only. The general expression reads

$$
Q_k^a + \frac{1}{g}\,\delta_{k1}\,(O_4)^a{}_2\,\frac{p_1}{P_{11}}\sqrt{1-\left(\frac{p_3}{p_1}\right)^2} - \frac{1}{g}\,\delta_{k2}\,\delta^a{}_2\,\frac{p_3}{P_{23}} = \Omega^a{}_b A_k^b - \frac{1}{2g}\,\varepsilon_{bc}{}^a\left(\Omega\,\partial_k\Omega^T\right)^{cb}.
$$

This equation determines the original gauge potential A_k^a as a function of the variables (37), provided that we use equations (27), (36), (29) and (33) to define those components Q_k^a that are not regarded as free variables. In the same way we can invert the momentum transformation equation (38h), taking into account the definitions (21) and (32). The result is

$$
\begin{aligned}
A_k^a &= \left(\Omega^T\right)^a{}_b\left(Q_k^b + \frac{1}{g}\,\delta_{k1}\,(O_4)^b{}_2\,\frac{p_1}{P_{11}}\sqrt{1-\left(\frac{p_3}{p_1}\right)^2} - \frac{1}{g}\,\delta_{k2}\,\delta^b{}_2\,\frac{p_3}{P_{23}}\right) \\
&\quad - \frac{1}{2g}\,\varepsilon_{bc}{}^a\left(\Omega^T\,\partial_k\Omega\right)^{cb} \qquad (40a) \\
\Pi_{ka} &= \left(\Omega^T\right)_a{}^b\,P_{kb}, \qquad (40b)
\end{aligned}
$$

where

$$
\begin{aligned}
Q_1^2 &= -\frac{1}{g\,P_{11}}\sum_{k=2}^{3}\left(\partial_k\,P_{k3} - g\,\varepsilon_b{}^c{}_3 Q_k^b\,P_{kc}\right) + \frac{1}{g}\,p_3\,\frac{P_{21}}{P_{11}P_{23}} \\
Q_1^3 &= \frac{1}{g\,P_{11}}\left(\partial_3\,P_{32} - \sum_{k=2}^{3} g\,\varepsilon_b{}^c{}_2 Q_k^b\,P_{kc}\right) \qquad (41) \\
Q_2^2 &= \frac{1}{g\,P_{23}}\left(\partial^k\,P_{k1} - g\,\varepsilon_b{}^c{}_1\,Q_3^b\,P_{3c}\right) \\
P_{12} &= P_{13} = P_{22} = 0
\end{aligned}
$$

and Ω^T is expressed in the variables (q_i, p_i), i.e.,

$$
\begin{aligned}
\Omega^T &= \begin{pmatrix}
\sqrt{1-(\frac{p_2}{p_1})^2}\cos(g\,q_2) & \frac{p_2}{p_1}\cos(g\,q_2) & \sin(g\,q_2) \\
-\sqrt{1-(\frac{p_2}{p_1})^2}\sin(g\,q_2) & -\frac{p_2}{p_1}\sin(g\,q_2) & \cos(g\,q_2) \\
\frac{p_2}{p_1} & -\sqrt{1-(\frac{p_2}{p_1})^2} & 0
\end{pmatrix} \\
&\quad \cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos(g\,q_1) & \sin(g\,q_1) \\
0 & -\sin(g\,q_1) & \cos(g\,q_1)
\end{pmatrix}
\end{aligned}
$$

$$
\cdot \begin{pmatrix} \frac{p_3}{p_1} & 0 & \sqrt{1-(\frac{p_3}{p_1})^2} \\ 0 & -1 & 0 \\ \sqrt{1-(\frac{p_3}{p_1})^2} & 0 & -\frac{p_3}{p_1} \end{pmatrix}
\cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(g\,q_3) & \sin(g\,q_3) \\ 0 & -\sin(g\,q_3) & \cos(g\,q_3) \end{pmatrix} .
$$

The transformation equations can also be obtained from a generating functional of the form

$$
F\big[p_1, q_2, p_3, \{Q_k^{a'}\}, \{\Pi_{ka}\}\big] = \int \mathcal{F}(\mathbf{x})\, d^3\mathbf{x}, \tag{42}
$$

where

$$
\begin{aligned}
\mathcal{F} \;=\; & \frac{2}{g}\,\eta\, p_1 \arctan\left(\frac{\sqrt{1-(\frac{p_3}{p_1})^2}\,||\Pi_1|| - [\Pi_{11}\sin(g\,q_2) + \Pi_{12}\cos(g\,q_2)]}{\sqrt{\Pi_{13}^2 - (\frac{p_3}{p_1})^2\,||\Pi_1||^2 + [\Pi_{11}\cos(g\,q_2) - \Pi_{12}\sin(g\,q_2)]^2}}\right) \\
& + \frac{2}{g}\,p_3 \arctan\Bigg(\Bigg[||\Pi_1||\;[N_1\sin(g\,q_2) + N_2\cos(g\,q_2)]\,\sqrt{1-(p_3/p_1)^2} \\
& \qquad + \eta\,||N||\left[\Pi_{13}^2 - \left(\frac{p_3}{p_1}\right)^2||\Pi_1||^2 + [\Pi_{11}\cos(g\,q_2) - \Pi_{12}\sin(g\,q_2)]^2\right]^{1/2}\Bigg] \\
& \qquad \times\Bigg[\left(\sqrt{1-(p_3/p_1)^2}\,K_1 - \frac{p_3}{p_1}||N||\,\Pi_{11}\right)\sin(g\,q_2) \\
& \qquad + \left(\sqrt{1-(p_3/p_1)^2}\,K_2 - \frac{p_3}{p_1}||N||\,\Pi_{12}\right)\cos(g\,q_2)\Bigg] \Bigg/ \\
& \qquad \Bigg[-(K_1^2 + K_2^2) + \left(\frac{p_3}{p_1}\right)^2||\Pi_1||^2 N_3^2 + [K_1\cos(g\,q_2) - K_2\sin(g\,q_2)]^2 \\
& \qquad + \left(\frac{p_3}{p_1}\right)^2||\Pi_1||^2\,[N_1\cos(g\,q_2) - N_2\sin(g\,q_2)]^2\Bigg]\Bigg) \\
& - Q^{ka'}\,\Omega_{a'}{}^{b}\,\Pi_{kb} + \frac{1}{2g}\,\varepsilon_{bc}{}^{a}\left(\Omega^T\,\partial_3\,\Omega\right)^{cb}\Pi_{3a} - \frac{1}{2g}\Pi_{11}N_1\,\partial_2\left(\frac{1}{\Pi_{12}^2 + \Pi_{13}^2}\right) \\
& + \frac{1}{g}\,\frac{\Pi_{21}(\Pi_{13}\,\partial_2\Pi_{12} - \Pi_{12}\,\partial_2\Pi_{13}) + \Pi_{11}(\Pi_{13}\,\partial_1\Pi_{12} - \Pi_{12}\,\partial_1\Pi_{13})}{\Pi_{12}^2 + \Pi_{13}^2} \\
& - \frac{1}{g}\left[\partial_1\,||\Pi_1|| + \partial_2\left(||\Pi_1||\,\frac{\Pi_{12}\Pi_{22} + \Pi_{13}\Pi_{23}}{\Pi_{12}^2 + \Pi_{13}^2}\right)\right]\arctan\left(\frac{K_1}{||\Pi_1||N_1}\right) \\
& + \frac{1}{2g}\left[\partial_2\left(\frac{\Pi_{11}N_1}{\Pi_{12}^2 + \Pi_{13}^2}\right)\right]\log\left(\frac{||N||^2}{M^8}\right)
\end{aligned}
$$

and

$$
K_a = \varepsilon_a{}^{bc}\,\Pi_{1b}N_c = \left(\delta_a{}^{c}\delta^{bd} - \delta_a{}^{d}\delta^{bc}\right)\Pi_{1b}\Pi_{1c}\Pi_{2d}.
$$

The components N_a are those defined in equation (39) and M denotes a constant with the dimension of energy. There is also a real phase η which can take the values ± 1. Expression (39) is used for the matrix Ω, and the primed index a' stands as a reminder

of the fact that only independent components $Q_k^{a'}$, i.e. those included in the list (37) should be summed over. Now the transformation equations read

$$
\begin{aligned}
q_1(\mathbf{x}) &= \frac{\delta F}{\delta p_1(\mathbf{x})} && (43a)\\
p_2(\mathbf{x}) &= -\frac{\delta F}{\delta q_2(\mathbf{x})} && (43b)\\
q_3(\mathbf{x}) &= \frac{\delta F}{\delta p_3(\mathbf{x})} && (43c)\\
P_{ka'}(\mathbf{x}) &= -\frac{\delta F}{\delta Q^{ka'}(\mathbf{x})} && (43d)\\
A_k^a(\mathbf{x}) &= -\frac{\delta F}{\delta \Pi_a^k(\mathbf{x})}. && (43e)
\end{aligned}
$$

Equations (43a) – (43c) reproduce equations (38a), (38e) and (38c) in a form where the components G_a are expressed in the variables $\{p_1, q_2, p_3, \Pi_{1a}\}$ by inverting equations (38d), (38b) and (38f), i.e.,

$$
\begin{aligned}
G_1 &= \Bigg(\frac{p_3}{p_1}\,||\Pi_1||\,\left[\Pi_{11}\cos(g\,q_2) - \Pi_{12}\sin(g\,q_2)\right]\\
&\qquad -\eta\,\Pi_{13}\sqrt{\Pi_{13}^2 - \left(\frac{p_3}{p_1}\right)^2 ||\Pi_1||^2 + \left[\Pi_{11}\cos(g\,q_2) - \Pi_{12}\sin(g\,q_2)\right]^2}\Bigg)\\
&\quad \times \frac{p_1\cos(g\,q_2)}{\Pi_{13}^2 + \left[\Pi_{11}\cos(g\,q_2) - \Pi_{12}\sin(g\,q_2)\right]^2}\\
G_2 &= -\Bigg(\frac{p_3}{p_1}\,||\Pi_1||\,\left[\Pi_{11}\cos(g\,q_2) - \Pi_{12}\sin(g\,q_2)\right]\\
&\qquad -\eta\,\Pi_{13}\sqrt{\Pi_{13}^2 - \left(\frac{p_3}{p_1}\right)^2 ||\Pi_1||^2 + \left[\Pi_{11}\cos(g\,q_2) - \Pi_{12}\sin(g\,q_2)\right]^2}\Bigg)\\
&\quad \times \frac{p_1\sin(g\,q_2)}{\Pi_{13}^2 + \left[\Pi_{11}\cos(g\,q_2) - \Pi_{12}\sin(g\,q_2)\right]^2} \qquad (44)\\
G_3 &= \Bigg(\frac{p_3}{p_1}\,||\Pi_1||\,\Pi_{13} + \eta\left[\Pi_{11}\cos(g\,q_2) - \Pi_{12}\sin(g\,q_2)\right]\\
&\qquad \times\sqrt{\Pi_{13}^2 - \left(\frac{p_3}{p_1}\right)^2 ||\Pi_1||^2 + \left[\Pi_{11}\cos(g\,q_2) - \Pi_{12}\sin(g\,q_2)\right]^2}\Bigg)\\
&\quad \times \frac{p_1}{\Pi_{13}^2 + \left[\Pi_{11}\cos(g\,q_2) - \Pi_{12}\sin(g\,q_2)\right]^2},
\end{aligned}
$$

where the sign η must be chosen so that

$$
\begin{aligned}
&\Bigg(\frac{p_3}{p_1}\,||\Pi_1||\,\left[\Pi_{11}\cos(g\,q_2) - \Pi_{12}\sin(g\,q_2)\right]\\
&\quad -\eta\,\Pi_{13}\sqrt{\Pi_{13}^2 - \left(\frac{p_3}{p_1}\right)^2 ||\Pi_1||^2 + \left[\Pi_{11}\cos(g\,q_2) - \Pi_{12}\sin(g\,q_2)\right]^2}\Bigg) \geq 0.
\end{aligned}
$$

Equations (43d) and (38h) are equivalent, and with the help of equations (38c), (38h) and (44) it is also possible to see the equivalence of equations (43e) and (40a). Although the generating functional looks rather complicated, its mere existence is sufficient to confirm that the transformation (38) is canonical. We have now all the necessary tools at hand for constructing the physical Hamiltonian.

3 Physical variables

The greatest advantage in passing to the new variables (37) is the fact that their behaviour in the limit $G_a \to 0$ is relatively simple to analyse. Of course, if we were to be exact, we would have to specify this limit precisely by starting from equation (2) and then defining suitable norms and function spaces for the fields A_k^a and Π_{ka}. Instead of doing so I will adopt a physicist's point of view and assume that it does not matter much which particular function spaces we use if our fields are sufficiently smooth and vanish rapidly enough at infinity. Looking at equations (38d) – (38f) we see then that Gauss's law is implemented in the new variables by setting

$$p_1 = p_2 = p_3 = 0. \tag{45}$$

That these constraints are preserved in time in the dynamics described by the Hamiltonian (1) is evident because p_1 and p_2 are constants of motion and $\dot{p}_3$ is proportional to the Gauss law generators. Equations (38a) – (38c) reveal similarly that the angles q_1, q_2 and q_3 become ambiguous when $G_a \to 0$ and therefore we must discard these variables as nonphysical. The physical variables are then the pairs $(Q_k^{a'}, P_{ka'})$, as their defining equations (38g) and (38h) are independent of G_a. Since the generating functional (42) does not contain explicit time dependence, the dynamics of $Q_k^{a'}$ and $P_{ka'}$ is governed by the Hamiltonian (1) under the constraint (45), i.e.,

$$H_{\text{phys}} = H_{|\,p_1 = p_2 = p_3 = 0}.$$

A formula for the Hamiltonian (1) in the variables (37) is most easily obtained with the help of equations (40). Since the Hamiltonian is invariant under gauge transformations of this form, we immediately get the result

$$H = \int \left(\frac{1}{2} \, P_{ka} P^{ka} + \frac{1}{4} \, \widetilde{\Phi}^a_{kl} \, \widetilde{\Phi}^{kl}_a \right) d^3\mathbf{x},$$

where

$$\begin{aligned} \widetilde{\Phi}^a_{kl} &= \partial_l \widetilde{Q}^a_k - \partial_k \widetilde{Q}^a_l + g \, \varepsilon_{bc}{}^a \widetilde{Q}^b_k \, \widetilde{Q}^c_l, \\ \widetilde{Q}^a_k &= Q^a_k + \frac{1}{g} \, \delta_{k1} \, (O_4)^a{}_2 \frac{p_1}{P_{11}} \sqrt{1 - \left(\frac{p_3}{p_1}\right)^2} - \frac{1}{g} \, \delta_{k2} \, \delta^a{}_2 \frac{p_3}{P_{23}} \end{aligned}$$

and the definitions (41) are implied. Imposing the constraint (45) it is then easy to see that

$$H_{\text{phys}} = \int \left(\frac{1}{2} \, P_{ka} P^{ka} + \frac{1}{4} \, \Phi^a_{kl} \, \Phi^{kl}_a \right) d^3\mathbf{x}, \tag{46}$$

where

$$\Phi^a_{kl} = \partial_l Q^a_k - \partial_k Q^a_l + g \, \varepsilon_{bc}{}^a Q^b_k \, Q^c_l$$

and

$$
\begin{aligned}
Q_1^2 &= -\frac{1}{g\,P_{11}} \sum_{k=2}^{3} \left(\partial_k\, P_{k3} - g\,\varepsilon_b{}^c{}_3 Q_k^b\, P_{kc}\right) \\
Q_1^3 &= \frac{1}{g\,P_{11}} \Big(\partial_3\, P_{32} - \sum_{k=2}^{3} g\,\varepsilon_b{}^c{}_2 Q_k^b\, P_{kc}\Big) \\
Q_2^2 &= \frac{1}{g\,P_{23}} \left(\partial^k\, P_{k1} - g\,\varepsilon_b{}^c{}_1\, Q_3^b\, P_{3c}\right) \\
P_{12} &= P_{13} \;=\; P_{22} \;=\; 0.
\end{aligned}
\tag{47}
$$

Equations (38h) and (39) also show that

$$
P_{11} \geq 0, \qquad P_{23} \geq 0.
$$

It may be a little surprising that the Hamiltonian (46) is local, because one would expect the Gauss law to produce nonlocal terms. However, the locality of H_{phys} becomes easy to understand if we look at the definitions (47). Our gauge choices have annihilated three momentum components, and when we solve Gauss's law for the coordinates conjugate to these momenta, the result is local. One should also note that the Hamiltonian density is singular at points where P_{11} or P_{23} vanishes. These are exactly the same points where the gauge transformation matrix (39) becomes ambiguous.

Now we would like to examine what the Hamiltonian (46) looks like at small and large values of the coupling constant g. For that purpose we note that every component Q_k^a consists of terms proportional to g^{-1} and terms independent of g. Therefore the field tensor components Φ_{kl}^a range from g^{-1} to g^1 and as a result, the Hamiltonian density takes the form

$$
\mathcal{H}_{\text{phys}} = \frac{1}{2g^2}\mathcal{H}^{(0)} + \frac{1}{g}\,\mathcal{H}^{(1)} + \mathcal{H}^{(2)} + g\,\mathcal{H}^{(3)} + \frac{g^2}{2}\mathcal{H}^{(4)}.
\tag{48}
$$

At small values of g the dominant term is $\mathcal{H}^{(0)}$, and it is rather straightforward to work out that

$$
\begin{aligned}
\mathcal{H}^{(0)} = {} & \left[-\partial_2\left(\frac{1}{P_{11}}\sum_{k=2}^{3}\partial_k\, P_{k3} + \frac{P_{21}}{P_{11}P_{23}}\sum_{k=1}^{3}\partial_k\, P_{k1}\right) - \partial_1\left(\frac{1}{P_{23}}\sum_{k=1}^{3}\partial_k\, P_{k1}\right)\right]^2 \\
& + \left[-\partial_3\left(\frac{1}{P_{11}}\sum_{k=2}^{3}\partial_k\, P_{k3} + \frac{P_{21}}{P_{11}P_{23}}\sum_{k=1}^{3}\partial_k\, P_{k1}\right)\right]^2 + \left[\partial_2\left(\frac{\partial_3 P_{32}}{P_{11}}\right)\right]^2 \\
& + \left[\partial_3\left(\frac{\partial_3 P_{32}}{P_{11}}\right)\right]^2 + \left[-\frac{\partial_3 P_{32}}{P_{11}P_{23}}\sum_{k=1}^{3}\partial_k\, P_{k1}\right]^2 + \left[\partial_3\left(\frac{1}{P_{23}}\sum_{k=1}^{3}\partial_k\, P_{k1}\right)\right]^2.
\end{aligned}
\tag{49}
$$

This expression looks a bit complicated, but it is noteworthy that $\mathcal{H}^{(0)}$ does not depend on the coordinates Q_k^a. At large values of g we similarly find the dominant term to be

$$
\begin{aligned}
\mathcal{H}^{(4)} \;=\; \Big\{ & \Big[\big(P_{33}Q_3^1 - P_{31}Q_3^3\big)\big(P_{33}Q_3^2 - P_{32}Q_3^3\big) + P_{23}Q_3^2\big(P_{33}Q_2^1 - P_{31}Q_2^3\big) \\
& + P_{23}P_{32}\big(Q_2^3 Q_3^1 - Q_2^1 Q_3^3\big)\Big]^2
\end{aligned}
$$

$$
\begin{aligned}
&+P_{23}^2\Big[-P_{11}Q_1^1Q_2^3+Q_2^1\big(P_{23}Q_2^1-P_{21}Q_2^3+P_{33}Q_3^1-P_{31}Q_3^3\big)\Big]^2\\
&+\Big[P_{23}Q_2^1\big(-P_{32}Q_3^1+P_{31}Q_3^2\big)-\big(P_{11}Q_1^1+P_{21}Q_2^1\big)\big(P_{33}Q_3^2-P_{32}Q_3^3\big)\Big]^2\\
&+\Big[-P_{23}Q_3^2\big(P_{23}Q_2^1-P_{21}Q_2^3+P_{33}Q_3^1\big)\\
&\quad+Q_3^3\big(P_{23}P_{32}Q_3^1+P_{21}P_{33}Q_3^2\big)-P_{21}P_{32}\big(Q_3^3\big)^2\Big]^2 \qquad (50)\\
&+P_{23}^2\Big[Q_3^1\big(P_{23}Q_2^1-P_{21}Q_2^3+P_{33}Q_3^1\big)-Q_3^3\big(P_{11}Q_1^1+P_{31}Q_3^1\big)\Big]^2\\
&+\Big[-P_{23}P_{32}\big(Q_3^1\big)^2+P_{23}Q_3^2\big(P_{11}Q_1^1+P_{31}Q_3^1\big)\\
&\quad+P_{21}Q_3^1\big(-P_{33}Q_3^2+P_{32}Q_3^3\big)\Big]^2\\
&+P_{11}^2\Big[-P_{23}Q_2^3Q_3^2-Q_3^3\big(P_{33}Q_3^2-P_{32}Q_3^3\big)\Big]^2\\
&+\big(P_{11}P_{23}\big)^2\big(Q_2^3Q_3^1-Q_2^1Q_3^3\big)^2\\
&+P_{11}^2\Big[P_{23}Q_2^1Q_3^2+Q_3^1\big(P_{33}Q_3^2-P_{32}Q_3^3\big)\Big]^2\Big\}\Big/\big(P_{11}P_{23}\big)^2.
\end{aligned}
$$

This is also a rather complicated expression, being fourth order in the coordinates Q_k^a and fractional in the momenta P_{ka}. Finally we should note that the form of the decomposition (48) is actually a matter of choice, since it is always possible to scale the variables by

$$(Q_k^a, P_{ka}) \to (g^{\alpha_{ka}} Q_k^a,\ g^{-\alpha_{ka}} P_{ka}),$$

where the α_{ka}'s are arbitrary constants. However, scalings like this would alter the g-dependence of the field tensor Φ_{kl}^a and the covariant derivative. As a result, the interpretation of g would also change. In the present form g is defined so that the limit $g \to 0$ corresponds to an Abelian theory. The singular behaviour of $\mathcal{H}_{\rm phys}$ in this limit then stems from an obvious qualitative difference between Abelian and non-Abelian theories in the function group method. Namely, the algebra (3) shows that for a non-Abelian theory ($g \neq 0$) the Gauss law generators must be parametrised with variables that contain one canonically conjugate pair, whereas in the Abelian case ($g = 0$) this parametrisation cannot contain canonical pairs at all. Also the solution of Gauss's law given in (47) is genuinely non-Abelian and impossible to extend to the Abelian case.

When quantising the Hamiltonian (46), we could try to quantise one of the limiting cases (49) or (50) first and then develop a perturbation expansion in appropriate powers of g. At least the weak coupling Hamiltonian (49), despite its complicated appearance, looks easy to quantise as its eigenstates would consist of common eigenstates of the momentum operators. The strong coupling Hamiltonian (50) is considerably more difficult to quantise in the canonical approach because we would have to solve problems connected with the ordering of operators and with the regularisation of higher order functional derivatives defined at the same point in space. Moreover, it is not clear whether large values of the bare coupling constant are physically relevant. As a general feature of quantisation one should also take into account that two classical systems connected by a

canonical transformation do not necessarily yield unitarily equivalent quantum systems. For example, in quantum mechanics it is often difficult to find a unitary transformation corresponding to action-angle variables in classical mechanics [18]. The fact that the transformation (38) is nonlinear might thus have an effect on the quantisation of the Hamiltonian (46). Finding a suitable quantisation procedure remains a problem to study.

4 Conclusions

The unconstrained Hamiltonian (46) lies at the end of a long journey which started from the temporal gauge Hamiltonian (1) and passed through the transformation (38), making it finally possible to implement Gauss's law in the new variables (37). The canonical pairs (q_i, p_i) turned out to be nonphysical, which led to the conclusion that the physical degrees of freedom are described by the gauge-invariant fields $(Q_k^{a'}, P_{ka'})$. Equation (47) then defined those components that are not free variables. The actual construction of the variables (37) relied on the parametrisation (6) and the complementary choices (18), (28) and (31). One could also easily experiment with different choices and derive alternative Hamiltonians corresponding to them by applying the general principles stated in section 2. In particular, the Poisson bracket relations (17) allow for a large variety of possible U(1) gauges, given only that the initial choice (7) is made.

An extension of this construction to more general Lie groups is relatively straightforward to outline. One should begin by deriving a parametrisation of the Gauss law generators similar to equation (6). Identifying the G_a's with elements of the corresponding Lie algebra, one should select the maximum number of new canonical momenta from the maximal Abelian subspace of the enveloping algebra. The remaining variables needed for the parametrisation should then be chosen so that the Lie algebra relations

$$\{G_a(\mathbf{x}), G_b(\mathbf{y})\} = -g\, f_{ab}{}^c\, G_c(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})$$

would hold as a consequence of the canonical Poisson brackets. After identifying which variables are gauge-dependent one should define gauge-invariant variables by transforming the gauge-dependent degrees of freedom away. The details of the construction would then depend on the Poisson brackets of the gauge-invariant variables and the way of defining those gauge degrees of freedom that are not fixed by the parametrisation of the Gauss law generators. No doubt that the calculations would be much more complicated than in the SU(2) case. In addition to generalising the Lie group, one could also extend the method by adding matter fields, in particular fermions, into the theory. However, it seems that the question of quantisation deserves the most attention in the future because it is crucial for the physical applicability of the Hamiltonian (46).

Acknowledgements

I am indebted to professor C. Cronström for suggesting this problem and for numerous discussions and comments. This work was partially supported by the Magnus Ehrnrooth foundation and the Finnish Cultural Foundation.

References

[1] J. Goldstone and R. Jackiw, Phys. Lett. B **74**, 81–84 (1978)

[2] V. Baluni and B. Grossman, Phys. Lett. B **78**, 226–230 (1978)

[3] A. G. Izergin, V. E. Korepin, M. A. Semenov-Tian-Shansky and L. D. Faddeev, Theor. Math. Phys. **38**, 1–9 (1979)

[4] M. Creutz, I. J. Muzinich and T. N. Tudron, Phys. Rev. D **19**, 531–539 (1979)

[5] M. B. Halpern, Phys. Rev. D **19**, 517–530 (1979)

[6] R. Marnelius, Nucl. Phys. B **194**, 433–444 (1982)

[7] F. Lenz, H. W. L. Naus and M. Thies, Annals Phys. **233**, 317–373 (1994)

[8] M. Bauer, D. Z. Freedman and P. E. Haagensen, Nucl. Phys. B **428**, 147–168 (1994), hep-th/9405028

[9] P. E. Haagensen and K. Johnson, Nucl. Phys. B **439**, 597–616 (1995), hep-th/9408164

[10] H. Nachbagauer, Phys. Rev. D **52**, 3672–3678 (1995), hep-ph/9503244

[11] L. Chen, M. Belloni and K. Haller, Phys. Rev. D **55**, 2347–2366 (1997), hep-ph/9609507

[12] H. Reinhardt, Nucl. Phys. B **503**, 505–529 (1997), hep-th/9702049

[13] A. M. Khvedelidze and H.-P. Pavel, Phys. Rev. D **59**, 105017 (1999), hep-th/9808102

[14] S. Lie, Math. Ann. **8**, 215–303 (1875)

[15] G. 't Hooft, Nucl. Phys. B **190**, 455–478 (1981)

[16] V. N. Gribov, Nucl. Phys. B **139**, 1–19 (1978)

[17] I. M. Singer, Commun. Math. Phys. **60**, 7–12 (1978)

[18] M. Moshinsky and T. H. Seligman, Annals Phys. **114**, 243–272 (1978)